Fermi-Dirac Statistics

Consider a system of N identical and indistinguishable fermions of half-integral spin. They obey Pauli exclusion principle. At a particular time the particles are distributed among the different energy states, so that n_1 particles have energy E_1, n_2 particles have energy E_2 and so on. Let us find out the total number of ways in which n_i particles can be distributed in g_i cells having the energy level E_i.

The first particle can be placed in any of the available g_i states i.e., this particle can be assigned to any one of the g_i sets of quantum number. Thus the particle can be distributed in g_i different ways. Similarly, the second particle can be arranged in $(g_i - 1)$ different ways and the process continues.

Thus the total number of different ways in which n_1 particles can be distributed among the available g_1 states with energy level E_1 is

$$g_1(g_1-1)(g_1-2)(g_1-3)\ldots\ldots\ldots\ldots[g_1-(n_1-1)]$$

$$=\frac{g_1(g_1-1)(g_1-2)(g_1-3)\ldots\ldots\ldots\ldots[g_1-(n_1-1)](g_1-n_1)!}{(g_1-n_1)!}$$

$$=\frac{g_1!}{(g_1-n_1)!} \qquad \textbf{(1)}$$

Further, since the particles are taken to be indistinguishable, it will not be possible to detect an difference if the n_1 particles are reshuffled into different states occupied by them in the energy level E_1. Therefore, the total number of distinguishable arrangement of n_1 particles in g_1 states is,

$$=\frac{g_1!}{n_1!(g_1-n_1)!} \qquad \textbf{(2)}$$

Therefore, the total number of different and distinguishable ways in which n_1, n_2, n_3 etc particles can be distributed among the various energy levels E_1, E_2, E_3 etc can be obtained by

$$P = \frac{g_1!}{n_1!(g_1 - n_1)!} \cdot \frac{g_2!}{n_2!(g_2 - n_2)!} \cdot \frac{g_3!}{n_3!(g_3 - n_3)!} \cdots\cdots\cdots$$

$$= \prod_{i=1}^{N} \frac{g_i!}{n_i!(g_i - n_i)!} \qquad \textbf{(3)}$$

$\prod_i$ **is called the product.**

When the particles is in equilibrium, the probability is maximum. When P is maximum logP is maximum in eq(3). Hence, the most probable distribution can be obtained by evaluating the maximum value of logP. This should also satisfy the two conditions that

$$N = \sum_i n_i = \text{Constant} \qquad E = \sum_i n_i E_i = \text{Constant} \qquad \textbf{(4)}$$

Taking logarithm of P, $\log P = \sum_{i}^{N} [\log_e g_i! - \log_e n_i! - \log_e (g_i - n_i)!]$

By Stirling's theorem, logx! = xlogx - x

$$\log P = \sum_{i}^{N} [(g_i \log_e g_i - g_i) - (n_i \log_e n_i - n_i) - \{(g_i - n_i)\log_e (g_i - n_i) - (g_i - n_i)] \quad \textbf{(6)}$$

Differential form of eq. (6) as follows:

$$-d(\log P) = [\sum_{i}^{N} \log_e n_i - \log_e (g_i - n_i)] dn_i$$

For most probable distribution (for maximum value): dlogP = 0

$$\sum_{i}^{N} \log_e n_i - \log_e (g_i - n_i)] dn_i = 0 \quad \textbf{(7)}$$

$$N = \sum_{i} n_i = \text{Constant} \Rightarrow \sum_{i} dn_i = 0 \quad \textbf{(8)}$$

$$E = \sum_{i} n_i E_i = \text{Constant} \Rightarrow \sum_{i} E_i dn_i = 0 \quad \textbf{(9)}$$

Eq. (8) and (9) can be incorporated into eq (7) by making use of Lagrange's method of undetermined multipliers. Multiplying eq (8) by α and eq (9) by β and adding to eq (7), we get

$$\sum_{i}^{N}[\log n_i - \log(g_i - n_i) + \alpha + \beta E_i]dn_i = 0$$

$$\Rightarrow [\log n_i - \log(g_i - n_i) + \alpha + \beta E_i]dn_i = 0$$

After integration with integrating cons. 0 (let).

$$[\log n_i - \log(g_i - n_i) + \alpha + \beta E_i] = 0$$

$$\Rightarrow \log\frac{n_i}{(g_i - n_i)} = -(\alpha + \beta E_i) \Rightarrow \frac{n_i}{(g_i - n_i)} = e^{-(\alpha + \beta E_i)}$$

$$\Rightarrow \frac{g_i - n_i}{n_i} = e^{(\alpha + \beta E_i)} \Rightarrow \frac{g_i}{n_i} = 1 + e^{(\alpha + \beta E_i)}$$

$$\Rightarrow n_i = \frac{g_i}{e^{\alpha + \beta E_i} + 1} \qquad \textbf{(10)}$$

Eg. (10) is known as Fermi-Dirac Distribution law.

Substituting $\beta = \dfrac{1}{kT}$

k: **Boltzmann constant**
T: **Temperature**

F-D distribution law in terms of temperature

$$n_i = \frac{g_i}{e^{(\alpha + E_i/kT)} + 1} \qquad \textbf{(11)}$$

In most cases α is negative and $\alpha = -\dfrac{E_F}{kT}$

Eq. (11) becomes

$$n_i = \frac{g_i}{e^{(E_i - E_F)/kT} + 1} \qquad \textbf{(12)}$$

The value of EF is positive and is independent of temperature. EF is called the Fermi energy and is defined as the maximum kinetic energy that a free electron can have at the absolute zero temperature.

Applications of Fermi-Dirac Statistics

Fermi–Dirac statistics has many applications in studying electrical and thermal conductivities, thermoelectricity, thermionic and photoelectric effects, specific heat of metals, etc. on the assumption that metals contain free electrons constituting like a perfect gas known as electron gas.