Bose-Einstein Statistics

Consider a system of N identical and indistinguishable bosons of integral spin. They do not obey Pauli exclusion principle. At a particular time the particles are distributed among the different energy states, so that n_1 particles have energy E_1, n_2 particles have energy E_2 and so on.

Let us find out the total number of ways in which n_i particles can be distributed in g_i cells (or states) having the energy level E_i. We have g_i items of one kind and n_i items of another kind.

We know that if we have A items of one kind and B items of another kind. **How many ways you arrange this items?** Since A and B are identical and indistinguishable, A and B can be arranged by themselves.

A_1 A_2 A_3 A_4 A_5 B_1 B_2 B_3 B_4

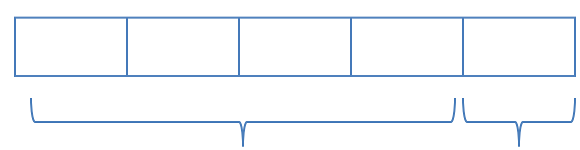
$$p = \frac{(A+B)!}{A!B!}$$

We have g_i items of one kind and n_i items of another kind. Apparently, we can say it would be

$$p = \frac{(n_i + g_i)!}{n_i!g_i!}$$

But not possible. Because particles can not be located outside the cells (or states)

States $g_i = 5$
Partitions $(g_i - 1) = 4$



Partitions $(g_i - 1)$ Particles n_i

The total number of possible distributions of the particles is given by the simultaneous permutations of n_i particles and (g_i-1) partitions and is given by $(n_i+g_i-1)!$ But this includes also the permutations of n_i particles among themselves as both these groups are internally indistinguishable. Thus the actual number of ways in which n_i particles can be distributed in g_i states is

$$= \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!} \qquad \textbf{(1)}$$

Further, since the particles are taken to be indistinguishable, it will not be possible to detect an difference if the n_1 particles are reshuffled into different states occupied by them in the energy level E_1. Therefore, the total number of distinguishable arrangement of n_1 particles in g_1 states is,

$$= \frac{(n_1 + g_1 - 1)!}{n_1!(g_1 - 1)!} \qquad \textbf{(2)}$$

Therefore, the total number of different and distinguishable ways in which n_1, n_2, n_3 etc particles can be distributed among the various energy levels E_1, E_2, E_3 etc can be obtained by

$$P = \frac{(n_1 + g_1 - 1)!}{n_1!(g_1 - 1)!} \cdot \frac{(n_2 + g_2 - 1)!}{n_2!(g_2 - 1)!} \cdot \frac{(n_3 + g_3 - 1)!}{n_3!(g_3 - 1)!} \cdots\cdots$$

$$= \prod_{i=1}^{N} \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!} \qquad \textbf{(3)}$$

$\prod_i$ **is called the product.**

When the particles is in equilibrium, the probability is maximum. When P is maximum logP is maximum in eq(3). Hence, the most probable distribution can be obtained by evaluating the maximum value of logP. This should also satisfy the two conditions that

$$N = \sum_i n_i = \text{Constant} \qquad E = \sum_i n_i E_i = \text{Constant} \qquad \textbf{(4)}$$

Taking logarithm of P, $\log P = \sum_i^{N} [\log_e (n_i + g_i - 1)! - \log_e n_i! - \log_e (g_i - 1)!]$

By Stirling's theorem, logx! = xlogx - x

$$\log P = \sum_{i}^{N}[(n_i + g_i - 1)\log(n_i + g_i - 1) - (n_i + g_i - 1) - n_i \log_e n_i + n_i - (g_i - 1)\log_e(g_i - 1) + (g_i - 1)]$$

$$= \sum_{i}^{N}[(n_i + g_i - 1)\log(n_i + g_i - 1) - n_i \log_e n_i - (g_i - 1)\log_e(g_i - 1)] \tag{6}$$

Differential form of eq. (6) as follows:

$$d(\log P) = \sum_{i}^{N}[d\{(n_i + g_i - 1)\log_e(n_i + g_i - 1)\} - d\{n_i \log_e n_i - 0]$$

$$= \sum_{i}^{N}[\log_e(n_i + g_i - 1)dn_i - \log_e n_i dn_i]$$

$$-d(\log P) = \sum_{i}^{N}[-\log_e(n_i + g_i - 1) + \log_e n_i]dn_i$$

For most probable distribution (for maximum value): dlogP = 0

$$\sum_{i}^{N}[-\log_e(n_i + g_i - 1) + \log_e n_i]dn_i = 0 \tag{7}$$

$$N = \sum_i n_i = \text{Constant} \Rightarrow \sum_i dn_i = 0 \quad \textbf{(8)}$$

$$E = \sum_i n_i E_i = \text{Constant} \Rightarrow \sum_i E_i dn_i = 0 \quad \textbf{(9)}$$

Eq. (8) and (9) can be incorporated into eq (7) by making use of Lagrange's method of undetermined multipliers. Multiplying eq (8) by α and eq (9) by β and adding to eq (7), we get

$$\sum_i^N [-\log(\mathrm{n_i} + \mathrm{g_i} - 1) + \log \mathrm{n_i} + \alpha + \beta \mathrm{E_i}]\mathrm{dn_i} = 0$$

$$\Rightarrow [-\log(\mathrm{n_i} + \mathrm{g_i} - 1) + \log \mathrm{n_i} + \alpha + \beta \mathrm{E_i}]\mathrm{dn_i} = 0$$

After integration with integrating cons. 0 (let).

$$[-\log(\mathrm{n_i} + \mathrm{g_i} - 1) + \log \mathrm{n_i} + \alpha + \beta \mathrm{E_i}] = 0$$

$$\Rightarrow [-\log(\mathrm{n_i} + \mathrm{g_i}) + \log \mathrm{n_i} + \alpha + \beta \mathrm{E_i}]$$

n_i +g_i is >>1, so we neglect 1.

$$\Rightarrow \log \frac{n_i}{(n_i + g_i)} = -\alpha - \beta E_i$$

$$\Rightarrow 1 + \frac{g_i}{n_i} = e^{\alpha + \beta E_i} \Rightarrow n_i = \frac{g_i}{e^{\alpha + \beta E_i} - 1} \qquad \textbf{(10)}$$

Eg. (10) is known as Bose-Einstein Distribution law.

Substituting $\beta = \frac{1}{kT}$

***k*: Boltzmann constant**
***T*: Temperature**

$$n_i = \frac{g_i}{e^{(\alpha + E_i / kT)} - 1} \qquad \textbf{(11)}$$

B-E distribution law in terms of temperature

Bose-Einstein Condensation (BEC)

According to the Bose-Einstein statistics particles are distributed at several states in the system. At the temperature drops below Tc, a larger fraction "condenses" into the ground state. As the temperature approaches absolute zero, all the particles condense into the ground state. This phenomenon, which occur even in the absence of direct interaction between the bosons, is called Bose-Einstein condensation.

Note that the fraction of particles that are condensing into the ground state are all in the same quantum state.

This state was first predicted, generally, in 1924–25 by Satyendra Nath Bose and Albert Einstein.
Experimentally proved in 1995 by Cornell and Wieman (University of Colorado)

Satyendra Nath Bose first sent a paper to Einstein on the quantum statistics of light quanta (now called photons), in which he derived Planck's quantum radiation law without any reference to classical physics. Einstein was impressed, translated the paper himself from English to German and submitted it for Bose to the *Zeitschrift für Physik*, which published it in 1924. Einstein then extended Bose's ideas to matter in two other papers. The result of their efforts is the concept of a Bose gas, governed by Bose–Einstein statistics, which describes the statistical distribution of identical particles with integer spin, now called bosons. Bosons, which include the photon as well as atoms such as helium-4 (^{4}He), are allowed to share a quantum state. Einstein proposed that cooling bosonic atoms to a very low temperature would cause them to fall (or "condense") into the lowest accessible quantum state, resulting in a new form of matter.

In 1938 Fritz London proposed BEC as a mechanism for superfluidity in ^{4}He and superconductivity. On June 5, 1995 the first gaseous condensate was produced by Eric Cornell and Carl Wieman at the University of Colorado at BoulderNIST–JILA lab, in a gas of rubidium atoms cooled to 170 nanokelvins (nK). Shortly thereafter, Wolfgang Ketterle at MIT demonstrated important BEC properties. For their achievements Cornell, Wieman, and Ketterle received the 2001 Nobel Prize in Physics. Many isotopes were soon condensed, then molecules, quasi-particles, and photons in 2010.