Differential equation of wave motion

A plane progressive wave is one which travels onward through the medium in a given direction without attenuation, i.e., with its amplitude constant.

$$y = a\sin\frac{2\pi}{\lambda}(vt - x) \qquad (1)$$

$$\therefore \frac{dy}{dt} = \frac{2\pi a v}{\lambda}\cos\frac{2\pi}{\lambda}(vt - x)$$

$$\Rightarrow \frac{d^2y}{dt^2} = -\frac{4\pi^2 v^2}{\lambda^2}a\sin\frac{2\pi}{\lambda}(vt - x)$$

$$\Rightarrow \frac{d^2y}{dt^2} = -\frac{4\pi^2 v^2}{\lambda^2}\cdot y \qquad (2)$$

Similarly, differentiating equation (1) with respect to x, we get the displacement curve

$$\therefore \frac{dy}{dx} = -\frac{2\pi}{\lambda} a \cos \frac{2\pi}{\lambda}(vt - x)$$

Differentiating the above expression again with respect to x, we get the rate of change of compression with distance

$$\therefore \frac{d^2y}{dx^2} = -\frac{4\pi^2}{\lambda^2} a \sin \frac{2\pi}{\lambda}(vt - x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{4\pi^2}{\lambda^2} \cdot y \qquad (3)$$

From relations (2) and (3), we have

$$\frac{d^2y}{dt^2} = v^2(-\frac{4\pi^2}{\lambda^2} \cdot y) = v^2 \frac{d^2y}{dx^2} \qquad (4)$$

Equation (4) is referred to as the differential equation of a plane or one-dimensional progressive wave. The general differential equation of wave motion can be written as

$$\frac{d^2y}{dt^2} = K\frac{d^2y}{dx^2} \qquad (5) \qquad \text{where,} \quad v = \sqrt{K}$$

Any equation of this form can unhesitatingly be declared to represent a plane progressive harmonic wave, the velocity of which is given by the square root of the co-efficient of dy^2/dx^2.

Now dy^2/dx^2 gives the rate of change of compression with distance, i.e., the curvature of the displacement curve. Hence, the differential equation as given by equation (5) may be interpreted to mean that

Particle acceleration at a point $[\frac{d^2y}{dt^2}]$ **= Wave velocity^2**

$[v^2]$ x Curvature of the displacement curve at the point $[\frac{d^2y}{dx^2}]$

Particle velocity and Wave velocity

A plane progressive wave is one which travels onward through the medium in a given direction without attenuation, i.e., with its amplitude constant.

$$y = a\sin\frac{2\pi}{\lambda}(vt - x) \qquad (1)$$

y= displacement of a particle of the medium at a distance x from the origin and at an instant of time t.
a = amplitude
v= Wave (or) phase velocity

Differentiating equation (1) with respect to time, we get the particle velocity

$$U = \frac{dy}{dt} = \frac{2\pi a v}{\lambda}\cos\frac{2\pi}{\lambda}(vt - x) \qquad (2)$$

The maximum value of the particle velocity is

$$U_{\max} = \frac{2\pi a}{\lambda} \cdot v$$

Or. Maximum particle velocity = $\frac{2\pi a}{\lambda}$ **x Wave velocity**

The acceleration of the particle velocity is given by

$$\frac{d^2 y}{dt^2} = -\frac{4\pi^2 v^2}{\lambda^2} a \sin\frac{2\pi}{\lambda}(vt - x)$$

$$\Rightarrow \frac{d^2 y}{dt^2}_{\max} = -\frac{4\pi^2 v^2}{\lambda^2} \cdot a \qquad (3)$$

The minus sign indicates that the acceleration is directed towards its mean position.

Now, differentiating equation (1) with respect to x, we get the slope of the displacement curve (as referred to as strain or compression).

$$\therefore \frac{dy}{dx} = -\frac{2\pi}{\lambda} a \cos \frac{2\pi}{\lambda}(vt - x) \qquad (4)$$

From equation (2) and (4), we get

$$U = \frac{dy}{dt} = -v \cdot \frac{dy}{dx}$$

Thus, particle velocity at a point = - (Wave velocity) x (Slope of the displacement curve at that point)

Energy density of a plane progressive wave

In a progressive wave motion, the energy derived from the source is passed on from particle to particle, so that there is a regular transmission of energy across every section of the medium. The term energy density of a plane progressive wave means the total energy per unit volume of the medium through which the wave is passing.

The equation of a plane progressive wave

$$y = a\sin\frac{2\pi}{\lambda}(vt - x) \qquad (1)$$

Here, v is the wave velocity. The velocity of the particle,

$$U = \frac{dy}{dt} = \frac{2\pi v}{\lambda}a\cos\frac{2\pi}{\lambda}(vt - x) \qquad (2)$$

The acceleration of the particle,

$$\frac{d^2y}{dt^2} = -\frac{4\pi^2v^2}{\lambda^2} a\sin\frac{2\pi}{\lambda}(vt - x) = -\frac{4\pi^2v^2}{\lambda^2}\cdot y \qquad (3)$$

Kinetic energy per unit volume:

Let us consider unit volume of the medium in the form of an extremely thin element of the medium parallel to the wave front. Now, density is mass per unit volume and since unit volume is being considered here,

Mass of the element = ρ, density of the medium

K. E. per unit volume = ½ (mass)x(velocity)^2 = ½ ρv^2

$$= \frac{2\pi^2v^2\rho}{\lambda^2} a^2\cos^2\frac{2\pi}{\lambda}(vt - x) \qquad (4)$$

Potential energy per unit volume:

The potential energy = work done per unit volume for a small displacement = Force x displacement

= mass x acceleration x displacement

$$= \rho \cdot \frac{d^2 y}{dt^2} dy$$

$$= \rho \cdot \frac{4\pi^2 v^2}{\lambda^2} y \cdot dy$$

Thus, the total work done when the layer is displaced from 0 to y is

$$= \int_0^y \rho \cdot \frac{4\pi^2 v^2}{\lambda^2} y \cdot dy = \rho \cdot \frac{2\pi^2 v^2}{\lambda^2} y^2$$

$$= \frac{2\pi^2 v^2 \rho}{\lambda^2} \cdot a^2 \sin^2 \frac{2\pi}{\lambda}(vt - x) \qquad (5)$$

Thus, the total energy per unit volume of the medium or the energy density of the plane progressive wave,

E = K. E. + P. E.

$$= \frac{2\pi^2 v^2 \rho}{\lambda^2} \cdot a^2 = 2\pi^2 n^2 \rho a^2$$

Problem 1: A source of sound has amplitude of 0.25 cm and a frequency of 512 Hz. If the velocity of sound in air is 340 m/s and the density of air is 0.00129 gm/cm^3, what is the rate of flow of energy per square cm?

Sol: Rate of low of energy per square cm = Energy density x velocity

$$= 2\pi^2 n^2 \rho a^2 v$$

$$= 2\pi^2 (512)^2 (0.00129)(0.25)^2 (34000)$$

$$= 1.417 \times 10^7 \ \text{erg/cm}^2.\text{s} = 1417 \ \text{J/cm}^2.\text{s}$$

Problem 2: **A plane progressive wave train of frequency 400 Hz has a phase velocity of 480 m/s. (i) How far apart are two points 30° out of phase? (ii) What is the phase difference between two displacements a a given point at times 10^{-3} s apart?**

Sol: **The equation of a plane progressive wave**

$$y = a\sin\frac{2\pi}{\lambda}(vt - x)$$

(i) Phase difference between two points:

$$= \frac{2\pi}{\lambda}(vt - x_1) - \frac{2\pi}{\lambda}(vt - x_2)$$

$$= 2\pi n\frac{(x_2 - x_1)}{v}$$

Phase difference between two points is 30°= π/6 rad

$$\therefore 2\pi n \frac{(x_2 - x_1)}{v} = \frac{\pi}{6}$$

$$\Rightarrow (x_2 - x_1) = \frac{1}{6} \cdot \frac{480 \times 10^2}{2 \times 400} = 10 \text{ cm} = 0.1 \text{ m}$$

(ii) Phase difference between two times:

$$= \frac{2\pi}{\lambda}(vt_2 - x) - \frac{2\pi}{\lambda}(vt_1 - x)$$

$$= 2\pi n(t_2 - t_1)$$

Phase difference between two times is 10^{-3} s

Phase difference between two times

$$= 2\pi \cdot 400 \cdot 10^{-3} = 0.8\pi \text{ rad} = 144^0$$

Problem 3: **Which of the following are solutions of the one dimensional wave equations?**

$$(1)\ y = x^2 + v^2t^2 \quad (2)\ y = x^2 - v^2t^2 \quad (3)\ y = (x - vt)^2$$

$$(4)\ y = 7x - 10t \quad (5)\ y = 2\sin x \cos vt \quad (6)\ y = \sin 2x \cos vt$$

Sol: **One dimensional wave equation**

$$\frac{d^2y}{dt^2} = v^2 \frac{d^2y}{dx^2}$$

(1)

$$\frac{dy}{dt} = 2v^2t \therefore \frac{d^2y}{dt^2} = 2v^2$$

$$\frac{dy}{dx} = 2x \therefore \frac{d^2y}{dx^2} = 2$$

Therefore, $2v^2 = v^2 \cdot 2$

This is wave equation.

After solving we get
(2) Not
(3) Yes
(4) Yes
(5) Yes
(6) Not