Composition of two SHM at right angle to each other and having time period in the ratio 1:1

Let us consider two SHM acting at right angle to each other and have the same period as the following equations:

$$x = a\sin(\omega t + \alpha) ------(1)$$

$$y = b\sin\omega t --------(2)$$

It is clear that the amplitudes of the two SHM are different. The initial phase difference between them is α. From equ(2), we can write:

$$\sin\omega t = \frac{y}{b},\ \cos = \sqrt{1 - \frac{y^2}{b^2}}$$

From equ(1), we can write:

$$\frac{x}{a} = \sin\omega t\cos\alpha + \cos\omega t\sin\alpha$$

$$\Rightarrow \frac{x}{a} = \frac{y}{b}\cos\alpha + \sqrt{1-\frac{y^2}{b^2}}\sin\alpha$$

$$\Rightarrow \frac{x}{a} - \frac{y}{b}\cos\alpha = \sqrt{1-\frac{y^2}{b^2}}\sin\alpha$$

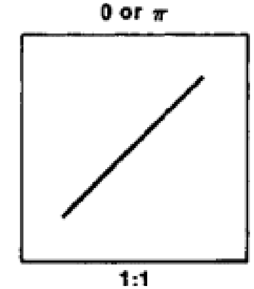
$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2}\cos^2\alpha - \frac{2xy}{ab}\cos\alpha = \left(1-\frac{y^2}{b^2}\right)\sin^2\alpha$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\alpha = \sin^2\alpha ------(3)$$

This represents the general equation of two SHM acting at right angle to each other. The resultant vibration of the particle will depend upon the value of α.

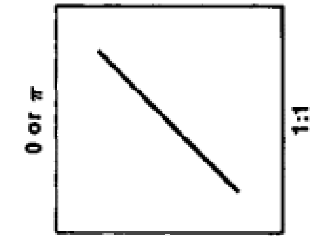
Special case:

(1) $\alpha = 0$ or 2π, eq(4) becomes $y = \frac{b}{a}x$



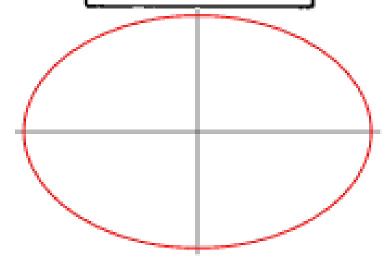
It is a straight line with positive slope.

(2) $\alpha = \pi$ $y = -\frac{b}{a}x$



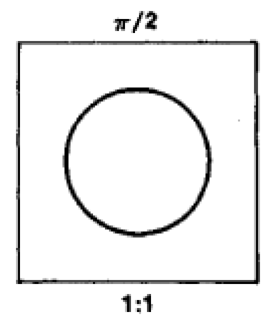
It is a straight line with negative slope.

(3) $\alpha = \pi/2$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



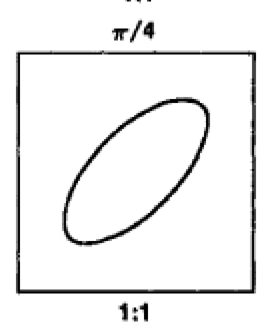
Equation for ellipse of axes *a* and *b*.

(4) $\alpha = \pi/2$, a =b $x^2 + y^2 = a^2$



Equation for circle of radius *a*.

(5) $\alpha = \pi/4$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab\sqrt{2}} = \frac{1}{2}$



Oblique ellipse of axes *a* and *b*.

Composition of two SHM at right angle to each other and having frequency ratio 2:1

Let us consider two SHM acting at right angle to each other and have the same period as the following equations:

$$x = a\sin(2\omega t + \alpha) -------(1)$$

$$y = b\sin\omega t ----------(2)$$

It is clear that the amplitudes of the two SHM are different. The initial phase difference between them is α. From equ(2), we can write:

$$\sin\omega t = \frac{y}{b},\ \cos = \sqrt{1 - \frac{y^2}{b^2}}$$

From equ(1), we can write:

$$\frac{x}{a} = \sin 2\omega t\cos\alpha + \cos 2\omega t\sin\alpha$$

$$\Rightarrow \frac{x}{a} = 2\sin\omega t\cos\omega t\cos\alpha + (1 - 2\sin^2\omega t)\sin\alpha$$

$$\Rightarrow \frac{x}{a} = \frac{2y}{b}\sqrt{1 - \frac{y^2}{b^2}}\cos\alpha + (1 - \frac{2y^2}{b^2})\sin\alpha$$

$$\Rightarrow \frac{x}{a} - (1 - \frac{2y^2}{b^2})\sin\alpha = \frac{2y}{b}\sqrt{1 - \frac{y^2}{b^2}}\cos\alpha$$

$$\Rightarrow [(\frac{x}{a} - \sin\alpha) + \frac{2y^2}{b^2}\sin\alpha]^2 = [\frac{2y}{b}\sqrt{1 - \frac{y^2}{b^2}}\cos\alpha]^2$$

$$\Rightarrow (\frac{x}{a} - \sin\alpha)^2 + 2(\frac{x}{a} - \sin\alpha)\frac{2y^2}{b^2}\sin\alpha + \frac{4y^4}{b^4}\sin^2\alpha = \frac{4y^2}{b^2}(1 - \frac{y^2}{b^2})\cos^2\alpha$$

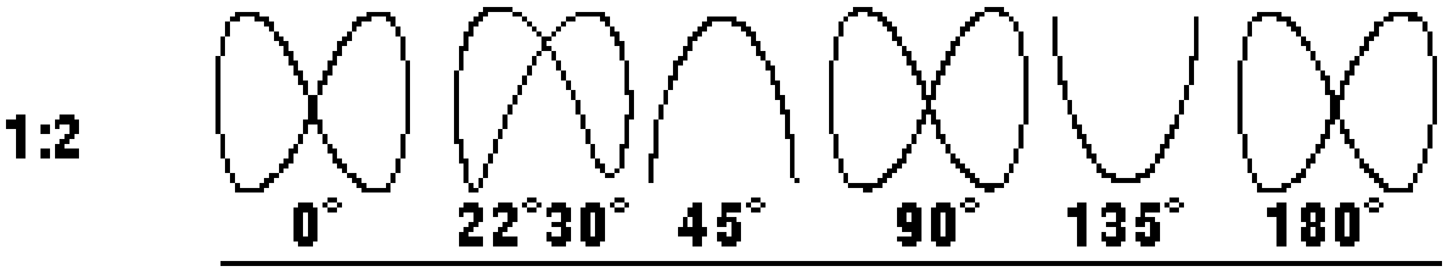
$$\Rightarrow (\frac{x}{a} - \sin\alpha)^2 + \frac{4y^4}{b^4}(\sin^2\alpha + \cos^2\alpha) - \frac{4y^2}{b^2}(\sin^2\alpha + \cos^2\alpha) + \frac{4y^2}{b^2}\frac{x}{a}\sin\alpha = 0$$

$$\Rightarrow (\frac{x}{a} - \sin\alpha)^2 + \frac{4y^4}{b^4} - \frac{4y^2}{b^2} + \frac{4y^2}{b^2}\frac{x}{a}\sin\alpha = 0$$

$$\Rightarrow \left(\frac{x}{a}-\sin\alpha\right)^2+\frac{4y^2}{b^2}\left(\frac{y^2}{b^2}+\frac{x}{a}\sin\alpha-1\right)=0 ----(3)$$

This represents the general equation of two SHM acting at right angle to each other having frequency ration 2:1

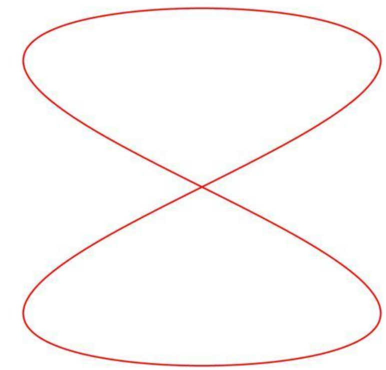
The following Lissajous figures contained two loops at different phase difference.



Special case:

(1) α = 0, π , 2π, ….etc, eq(3) becomes

$$\Rightarrow {\frac{x^2}{a^2}}^{2} + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2}-1\right)=0 ----(4)$$

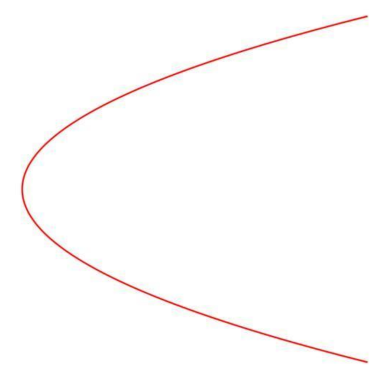


Equation of a two loops.

(2) α = π/2, eq(3) becomes

$$\Rightarrow \left(\frac{x}{a}-1\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2}+\frac{x}{a}-1\right)=0$$

$$\Rightarrow \left(\frac{x}{a}-1\right)^2 + \frac{4y^2}{b^2}\left(\frac{x}{a}-1\right)+\frac{4y^4}{b^4}=0$$



$$\Rightarrow \left[\left(\frac{x}{a}-1\right)^2 + \frac{2y^2}{b^2}\right]^2 = 0 \Rightarrow \left(\frac{x}{a}-1\right)^2 + \frac{2y^2}{b^2}=0$$

$$\Rightarrow y^2 = -\frac{b^2}{2a^2}(x-a) ----(5)$$

This represents the equation of a parabola with vertex at (a, 0).