Damped Oscillation

Every system inevitably has dissipative features through which the mechanical energy of the vibration is depleted. We shall now consider how the equation of free-vibration is modified by the introduction of dissipative forces. In this case, the resistive force is exerted oppositely to the direction velocity itself. The statement of Newton's law for the moving mass, m, can be written as:

$$m\frac{d^2y}{dt^2} = -bv - kx$$

$$\Rightarrow \frac{d^2y}{dt^2} = -\frac{b}{m}\frac{dy}{dt} - \frac{k}{m}x \qquad \lambda = \frac{b}{2m}$$

$$\Rightarrow \frac{d^2y}{dt^2} + 2\lambda\frac{dy}{dt} + \omega^2 y = 0 --- (1) \qquad \omega^2 = \frac{k}{m}$$

In this case, the damping is characterized by the quantity λ, having the dimension of frequency, and the constant ω would represent the angular frequency of the system if the damping is absent.

Equation (1) is the differential equation of a damped harmonic oscillator. This is the homogeneous linear type differential equation of the second order, must have at least on solution of the form:

$y = Ae^{pt}$ **where, A and p are both arbitrary constant.**

Let this be used as a trial solution. Differentiating $y = Ae^{pt}$ **with respect to time, we get,**

$$\frac{dy}{dt} = pAe^{pt}, \frac{d^2y}{dt^2} = p^2Ae^{pt}$$

Putting these values into the equation (1), we can write

$$p^2Ae^{pt} + 2\lambda pAe^{pt} + \omega^2 Ae^{pt} = 0$$

$$\Rightarrow p^2 + 2\lambda p + \omega^2 = 0 - - - - - - (2)$$

Equ (2) is clearly a quadratic equation in *p*, the solution of which is:

$$p = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 4\omega^2}}{2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

The differential equation (1) is, therefore, satisfies by two values of y,

$$y = Ae^{\left(-\lambda+\sqrt{\lambda^2-\omega^2}\right)t}$$

and $$y = Ae^{\left(-\lambda-\sqrt{\lambda^2-\omega^2}\right)t}$$

The equation being a linear one, the linear sum of the two linearly independent solution of the equation. Thus the general solution is:

$$y = A_1 e^{\left(-\lambda+\sqrt{\lambda^2-\omega^2}\right)t} + A_2 e^{\left(-\lambda-\sqrt{\lambda^2-\omega^2}\right)t} -----(3)$$

Where A_1 and A_2 are two arbitrary constants. The values of the arbitrary constants A_1 and A_2 may be determined as follows.

Differentiating equation (3) with respect to *t*, we get:

$$\frac{dy}{dt} = \left(-\lambda+\sqrt{\lambda^2-\omega^2}\right)A_1 e^{\left(-\lambda+\sqrt{\lambda^2-\omega^2}\right)t} + \left(-\lambda-\sqrt{\lambda^2-\omega^2}\right)A_2 e^{\left(-\lambda-\sqrt{\lambda^2-\omega^2}\right)t} --(4)$$

Let the maximum value of the displacement *y* be $y_{max}=a_0$ say at time *t* =0, Then we have from equation (3),

$$y_{\max} = a_0 = A_1 + A_2 \; - - - - (5)$$

Again, the velocity is zero at maximum displacement; $\frac{dy}{dt} = 0, \text{ at } t = 0$

Hence, from equation (4), we have

$$0 = \left(-\lambda + \sqrt{\lambda^2 - \omega^2}\right) A_1 + \left(-\lambda - \sqrt{\lambda^2 - \omega^2}\right) A_2$$

$$\Rightarrow -\lambda (A_1 + A_2) + \sqrt{\lambda^2 - \omega^2}\,(A_1 - A_2) = 0$$

$$\Rightarrow A_1 - A_2 = \frac{\lambda a_0}{\sqrt{\lambda^2 - \omega^2}} \; - - - - - (6)$$

Solving equation (5) and (6), we have

$$A_1 = \frac{a_0}{2}\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - \omega^2}}\right) \qquad A_2 = \frac{a_0}{2}\left(1 - \frac{\lambda}{\sqrt{\lambda^2 - \omega^2}}\right)$$

Substituting these values of A_1 and A_2 in equation (3), we have

$$y=\frac{a_0}{2}\left\{\left(1+\frac{\lambda}{\sqrt{\lambda^2-\omega^2}}\right)e^{\left(-\lambda+\sqrt{\lambda^2-\omega^2}\right)t}+\left(1-\frac{\lambda}{\sqrt{\lambda^2-\omega^2}}\right)e^{\left(-\lambda-\sqrt{\lambda^2-\omega^2}\right)t}\right\}\quad --(7)$$

This is the solution of a damped harmonic vibration.

Three important cases now arise: $(1)\ \lambda^2>\omega^2$ $(2)\ \lambda^2=\omega^2$ $(3)\ \lambda^2<\omega^2$