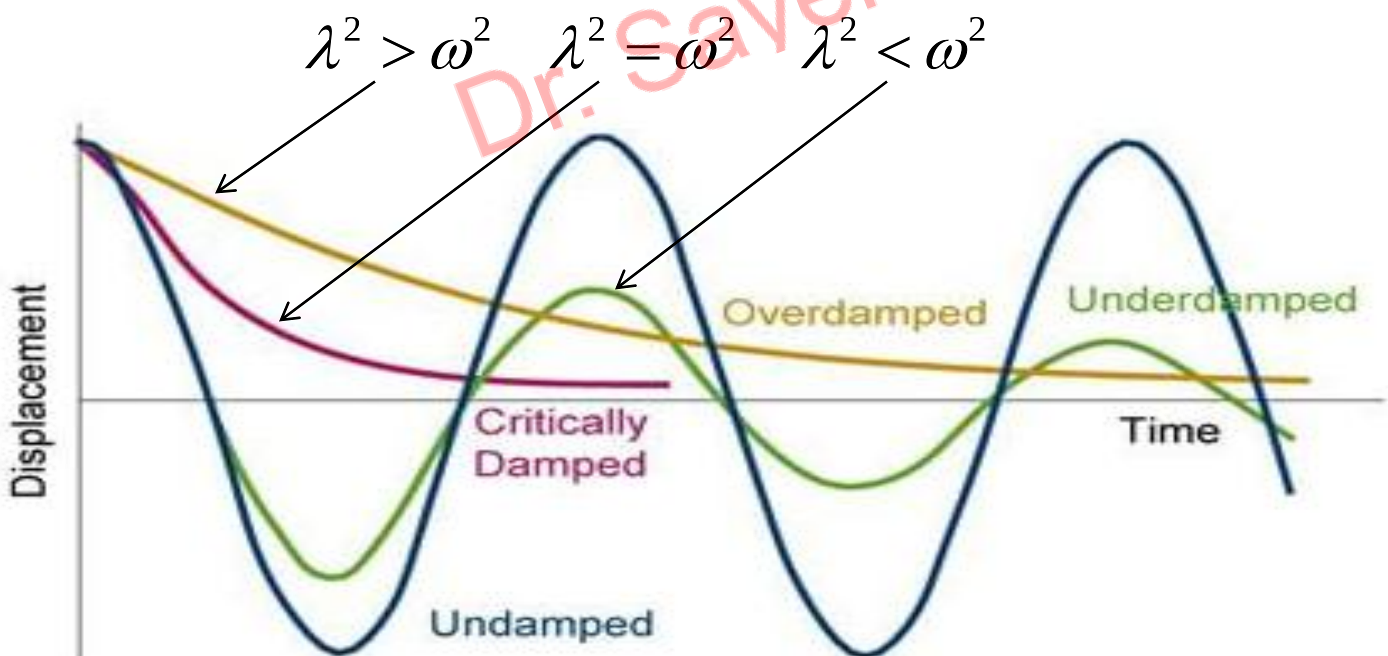
**The solution of a damped harmonic motion is therefore written as:**

$$y=\frac{a_0}{2}\left\{\left(1+\frac{\lambda}{\sqrt{\lambda^2-\omega^2}}\right)e^{\left(-\lambda+\sqrt{\lambda^2-\omega^2}\right)t}+\left(1-\frac{\lambda}{\sqrt{\lambda^2-\omega^2}}\right)e^{\left(-\lambda-\sqrt{\lambda^2-\omega^2}\right)t}\right\}--(7)$$

**Three important cases now arise:** $(1)\lambda^2>\omega^2$ $(2)$ $\lambda^2=\omega^2$ $(3)$ $\lambda^2<\omega^2$



**Over damped motion:** $\lambda^2 > \omega^2$ **which indicates that damping is large.**

$\sqrt{\lambda^2 - \omega^2}$ **is clearly a real quantity with a positive value, less than λ.**

**Thus, each of the two terms on the right side of equation (7) has an exponential term with a negative power and hence each decreases exponentially with time. In this case, the particle does not vibrate. The displacement, after attaining its maximum value falls off asymptotically to zero in the figure.**



**There is thus no oscillation and the motion is, therefore, called overdamped or aperiodic or dead beat. Examples are dead beat galvanometer or a pendulum oscillating in a viscous fluid like oil.**

**Critically damped motion:** $\lambda^2 = \omega^2$ **which indicates that damping is large.**

$\sqrt{\lambda^2 - \omega^2} = 0$ **so that, each of the two terms on the right hand side of equation (7) becomes infinite and the solution breaks down.**

**Let us, however, consider that $\lambda^2$ is not quite equal to $\omega^2$, but very nearly so, so that $\sqrt{\lambda^2 - \omega^2} = h$, a very small quantity but not zero.**



**Then, we have from eq(3)**

$$y = A_1 e^{(-\lambda+h)t} + A_2 e^{(-\lambda-h)t}$$

$$\Rightarrow y = e^{-\lambda t}\left(A_1 e^{ht} + A_2 e^{-ht}\right)$$

$$\Rightarrow y = e^{-\lambda t}\left\{A_1(1 + ht + \frac{h^2t^2}{2!} + \frac{h^3t^3}{3!} + \ldots) + A_2(1 - ht + \frac{h^2t^2}{2!} - \frac{h^3t^3}{3!} + \ldots)\right\}$$

**Neglecting terms containing second and higher powers of *h*, we get**

$$y = e^{-\lambda t}\left\{A_1(1+ht) + A_2(1-ht)\right\}$$

$$\Rightarrow y = e^{-\lambda t}\left\{(A_1 + A_2) + ht(A_1 - A_2)\right\} --- (8)$$

**Putting:** $A_1 + A_2 = M$ **and** $(A_1 - A_2)h = N$

**We have,** $\Rightarrow y = e^{-\lambda t}(M + Nt)$

**Recalling that** $y = y_{\max} = a_0$ and $\dfrac{dy}{dt} = 0$ at $t$=0, we have

$$M = y_{\max} = a_0 \text{ and } N = \lambda a_0$$

$$\Rightarrow y = e^{-\lambda t}(a_0 + \lambda a_0 t)$$

$$\Rightarrow y = a_0 e^{-\lambda t}(1 + \lambda t) ---- (9)$$

Second term $a_0 \lambda t e^{-\lambda t}$ decays less rapidly than the first term $a_0 e^{-\lambda t}$.

The displacement of the oscillator first increases but as *t* increases the exponential factor $e^{-\lambda t}$ becomes more important and the displacement decreases rapidly reaching the value zero for a finite value of *t*. The oscillator just ceases to oscillate and its motion just becomes aperiodic or non-oscillatory. This is called the case of critical damping.



This principle finds application in many pointer-type instruments like galvanometers where the pointer moves at once to and stays at, the correct position, without any annoying oscillations. It may be seem that although the motion is non-oscillatory both in case of critical damping and overdamping the time taken to reach near the equilibrium position from a given displacement becomes greater and greater with the increase of damping.

**Under damped motion:** $\lambda^2 < \omega^2$

**The quantity** $\sqrt{\lambda^2 - \omega^2}$ **is clearly imaginary, say equal to ig, where**

$g = \sqrt{\omega^2 - \lambda^2}$ **is a real quantity.**

**Then, we have from eq(3)** $y = A_1 e^{(-\lambda + ig)t} + A_2 e^{(-\lambda - ig)t}$

$$\Rightarrow y = e^{-\lambda t}\left(A_1 e^{igt} + A_2 e^{-igt}\right)$$

$$\Rightarrow y = e^{-\lambda t}\left\{A_1(\cos gt + i\sin gt) + A_2(\cos gt - i\sin gt)\right\}$$

$$\Rightarrow y = e^{-\lambda t}\left\{(A_1 + A_2)\cos gt + i(A_1 - A_2)\sin gt\right\}$$

**Putting:** $A_1 + A_2 = A$ **and** $i(A_1 - A_2) = B$

$$\Rightarrow y = e^{-\lambda t}\left\{A\cos gt + B\sin gt\right\} - - - (10)$$

**Equation (10) is the equation of a damped harmonic oscillator with amplitude $a_0 e^{-\lambda t}$ and frequency $= \sqrt{w^2 - \lambda^2} / 2\pi$**

**It is so called because the sine term in the equation suggests the oscillatory character of the motion and the exponential term, the gradual damping out of the oscillation in Fig.**



**The amplitude of the oscillation does not remain constant at a0, which represents the amplitude in the absence of any damping but decays exponentially with time to zero, in accordance with the term $e^{-\lambda t}$. The term $e^{-\lambda t}$ is called the damping factor.**