Forced Vibration

When an external periodic force is continuously applied to the oscillator, of a frequency not necessarily the same as the natural frequency of the oscillator, the oscillator ultimately follow to the driving force after some initial erratic movements. The oscillator settles down to oscillating with the frequency of the driving force and a constant amplitude and phase as long as the applied force remains operative. Such vibration of the oscillator is called forced vibration.

The amplitude of the oscillator depends on the difference between its natural frequency and the frequency of the driving force. The amplitude will be large if difference in frequencies is small and vice versa.

Let, the periodic force which is applied on a damped harmonic oscillator be $F = f_0 \sin pt$, which is obviously a sinusoidal force of amplitude f_0 and frequency *p*/2π.

The differential equation of the forced vibration can be written as follows:

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt} + f_0 \sin pt ----(1)$$

where, $-b\frac{dy}{dt}$ and $-ky$ are the damping and restoring forces on the oscillator, respectively.

$$\Rightarrow \frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{f_0}{m}\sin pt ----(2)$$

$$\Rightarrow \frac{d^2y}{dt^2} + 2\lambda\frac{dy}{dt} + w^2 y = F_0 \sin pt ----(3)$$

Let a particular solution of the equation of motion of the oscillator as given by equation (3), after the steady state has been attained, be

$$y = A\sin(pt - \phi) - - - - - - (4)$$

***A* is the amplitude of the forced vibration, ϕ is the phase difference between the driving force and the oscillator.**

$$\therefore \frac{dy}{dt} = Ap\cos(pt - \phi) - - - - - - (5)$$

$$\therefore \frac{d^2y}{dt^2} = -Ap^2\sin(pt - \phi) = -p^2 y - - - - (6)$$

Putting the values of equations (4-6) into the equation (3), and then we can obtain

$$-Ap^2\sin(pt - \phi) + 2\lambda Ap\cos(pt - \phi) + \omega^2 A\sin(pt - \phi) = F_0 \sin pt$$

$$\Rightarrow (\omega^2 - p^2)A\sin(pt - \phi) + 2\lambda Ap\cos(pt - \phi) = F_0 \sin\{(pt - \phi) + \phi\}$$

$$\Rightarrow (\omega^2 - p^2)A\sin(pt-\phi) + 2\lambda Ap\cos(pt-\phi)$$
$$= F_0\sin(pt-\phi)\cos\phi + F_0\cos(pt-\phi)\sin\phi \text{ --}(7)$$

Taking the coefficients of sin(*pt-*ϕ) and cos(*pt-*ϕ) in both sides of equation (7), we get

$$(\omega^2 - p^2)A = F_0\cos\phi \text{ ----}(8)$$

$$2\lambda Ap = F_0\sin\phi \text{ ----}(9)$$

Taking square on both sides of equations (8) and (9) and then add them, we get

$$(\omega^2 - p^2)^2A^2 + 4\lambda^2p^2A^2 = F_0^2$$

$$\therefore A = \frac{F_0}{\sqrt{(\omega^2-p^2)^2 + 4\lambda^2p^2}} \text{ ------}(10)$$

This is the equation for amplitude in forced vibration.

Again, divide equation (9) by (8) and get

$$\tan\phi = \frac{F_0 \sin\phi}{F_0 \cos\phi} = \frac{2\lambda A p}{A\left(\omega^2 - p^2\right)} = \frac{2\lambda p}{\left(\omega^2 - p^2\right)}$$

$$\therefore \phi = \tan^{-1}\left\{\frac{2\lambda p}{\left(\omega^2 - p^2\right)}\right\} ------(11)$$

Therefore, the solution of the forced vibration can be written as

$$y = A\sin(pt - \phi)$$

$$y = \frac{F_0}{\sqrt{(\omega^2 - p^2)^2 + 4\lambda^2 p^2}} \sin\left\{pt - \tan^{-1}\frac{2\lambda p}{\left(\omega^2 - p^2\right)}\right\} ------(12)$$

Equation (12) represents a simple harmonic motion of frequency $p/2\pi$, i.e., the same as that of the driving force, but lagging behind it in phase by $\phi = \tan^{-1}\left\{\frac{2\lambda p}{\left(\omega^2 - p^2\right)}\right\}$ **, where ϕ lies between 0 and π.**

Maximum amplitude of a driven oscillator
(Resonance condition)

When the damping, i.e., λ has a finite value greater than zero, the value of the amplitude, as given by the equation (12) will obviously be maximum when the denominator in the equation has its minimum value, i.e., when

$$\sqrt{(\omega^2 - p^2)^2 + 4\lambda^2 p^2} = 0$$

$$\Rightarrow (\omega^2 - p^2)^2 + 4\lambda^2 p^2 = 0$$

For maximum value, $$\therefore \frac{d}{dp}\left\{(\omega^2 - p^2)^2 + 4\lambda^2 p^2\right\} = 0$$

$$\Rightarrow -2(\omega^2 - p^2)2p + 4\lambda^2 2p = 0$$

$$\Rightarrow (\omega^2 - p^2) = 2\lambda^2$$

$$\Rightarrow p = \sqrt{\omega^2 - 2\lambda^2}$$

Thus, the amplitude will be maximum when the driving frequency

$$\frac{p}{2\pi} = \frac{\sqrt{\omega^2 - 2\lambda^2}}{2\pi}$$

Let us denote this particular value of the frequency by $\dfrac{p_r}{2\pi}$

This state of vibration when the amplitude of the oscillator becomes maximum is called resonance.

This particular forced frequency $\dfrac{p_r}{2\pi}$ for which resonance takes place is called resonant frequency.

Maximum amplitude, $A_{\max} = \dfrac{F_0}{\sqrt{(\omega^2 - p^2)^2 + 4\lambda^2 p^2}}$

$$= \frac{F_0}{\sqrt{\left\{\omega^2 - \left(\omega^2 - 2\lambda^2\right)\right\}^2 + 4\lambda^2\left(\omega^2 - 2\lambda^2\right)}}$$

$$= \frac{F_0}{\sqrt{4\lambda^4 + 4\lambda^2\omega^2 - 8\lambda^4}}$$

$$= \frac{F_0}{\sqrt{4\lambda^2\omega^2 - 4\lambda^4}}$$

$$\therefore A_{\max} = \frac{F_0}{\sqrt{4\lambda^2(\omega^2 - \lambda^2)}} = \frac{F_0}{2\lambda\sqrt{(\omega^2 - \lambda^2)}}$$

When the damping is very small (i.e, $\lambda \approx 0$), λ^2 can be neglecting, and therefore the maximum amplitude becomes,

$$A_{\max} = \frac{F_0}{2\lambda\omega}$$