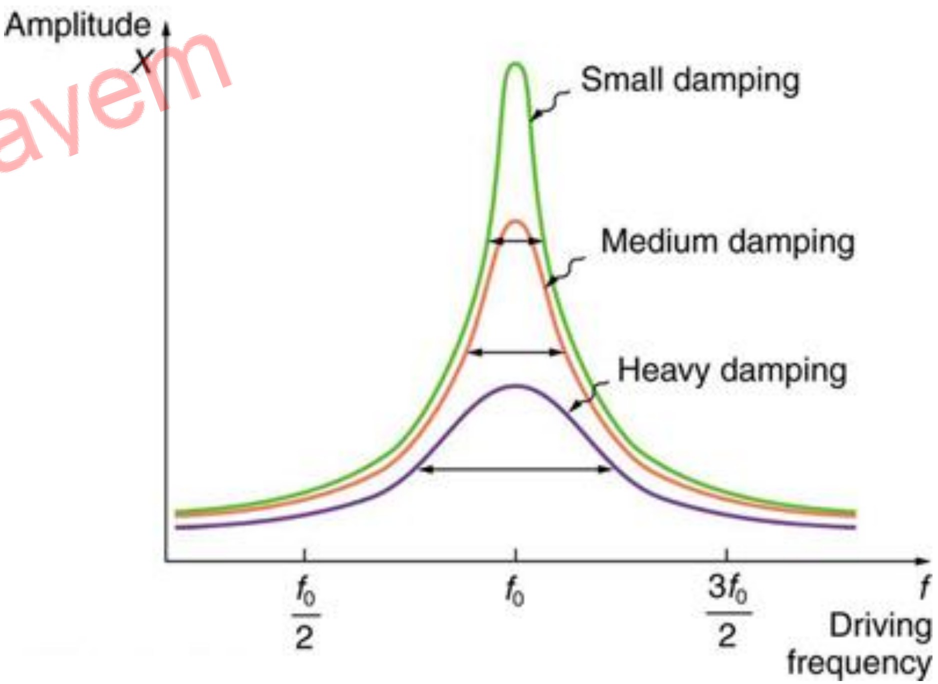
# Amplitude of forced vibration

**We know the amplitude of the forced vibration is as follows:**

$$A = \frac{F_0}{\sqrt{(\omega^2 - p^2)^2 + 4\lambda^2 p^2}}$$

**The amplitudes of the forced vibration is depends on the frequency of the driving force, *p*, and the damping factor, $\lambda$. The graph shows the amplitudes vs driving frequency at different damping factor. As the damping larger, the amplitudes became smaller.**



## Quality Factor (Q-factor)

**The Q-factor of an oscillator is defined as the ratio of the energy stored to the energy loss per cycle at resonance condition in the oscillator.**

$$Q = \frac{\omega}{2\lambda}$$

**If λ is minimum, Q became higher, and if λ =0, Q became infinite. Q became zero if the damping is max (infinite).**



## Sharpness of Resonance

**The sharpness of resonance may be regarded, in a way, as a measure of the rate of falls of amplitude from its maximum value at the resonant frequency, on either side of it. The sharper the fall in amplitude, the sharper the resonance. It can be further be seen that the smaller the damping, the sharper the resonance. In fact, it can be shown that the sharpness of resonance is inversely proportional to the square of the damping constant, λ.**

**Problem-1:** **A particle of mass 3 gm is subjected to an elastic constant 48 dyne-cm$^{-1}$ and a damping constant of 12 dyne-cm$^{-1}$sec. Justify that, the motion is oscillatory. In this case, find the time period of oscillation.**

## Solution:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{48}{3}} = 4\ \mathrm{s}^{-1}$$

$$2\lambda = \frac{b}{m} \Rightarrow \lambda = \frac{b}{2m} = \frac{12}{2\times 3} = 2\ \mathrm{s}^{-1}$$



$$\text{Since},\quad \lambda < \omega \ \text{(underdamped), The motion is oscillatory.}$$

**Frequency of the oscillation:** $$f = \frac{\sqrt{\omega^2 - \lambda^2}}{2\pi}$$

**Time period:** $$T = \frac{2\pi}{\sqrt{\omega^2 - \lambda^2}} = \frac{2\pi}{\sqrt{4^2 - 2^2}} = 1.81\ \mathrm{s}$$

**Problem-2:** A massless spring, suspended from a rigid support, carries a mass of 500 gm at its lower end and the system oscillate with a frequency of 5 Hz. If the amplitude is reduced to half its undamped value in 20 s, calculate (i) the force constant of the spring, (ii) the relaxation time of the system and (iii) its quality factor. (iv) What would be the quality factor of the system if the suspended mass be reduced to 150 gm?

## Solution:

**(i) The force constant k of the mass-spring system is given by**



$$\omega = \sqrt{\frac{k}{m}} \Rightarrow k = \omega^2 m$$

$$= (2\pi f)^2 m = (2\cdot\pi\cdot5)^2 (500) = 4.93\times10^5 \,\text{dyne/cm}$$

**(ii) The amplitude is reduced to half its undamped value in 20 s**

$$A = A_0 e^{-\lambda t}$$

$$\Rightarrow \frac{A_0}{2} = A_0 e^{-\lambda t}$$

$$\Rightarrow \frac{1}{2} = e^{-20\lambda}$$

$$\Rightarrow 20\lambda = \ln 2 \Rightarrow \lambda = \frac{0.693}{20} = 0.0347$$

**The relaxation time:** $$\tau = \frac{1}{2\lambda} = \frac{1}{2\times 0.0347} = 14.43 \text{ s}$$

**(iii) The quality factor:** $$Q = \frac{\omega}{2\lambda} = \frac{2\cdot\pi\cdot 5}{2\times 0.0347} = 452.7 \text{ s}$$



**(iii) The force constant k remains the same irrespective of the mass suspended from the spring and so does the damping factor b.**

$$2\lambda = \frac{b}{m}$$

$$\Rightarrow b = 2\lambda m = 2\times 0.0347\times 500 = 34.7$$

**When the mass is reduced to 150 gm, the frequency**

$$\omega^{/} = \sqrt{\frac{k}{m^{/}}} = \sqrt{\frac{4.93\times10^{5}}{150}} = 57.35 \ \ \text{Hz}$$

**Now the relaxation time:**

$$\tau^{/} = \frac{1}{2\lambda^{/}} = \frac{m^{/}}{b} = \frac{150}{34.7} = 4.32 \ \ \text{s}$$



**So the quality factor:**

$$Q = \frac{\omega^{/}}{2\lambda^{/}} = \omega^{/}\tau^{/} = 57.35\times4.32 = 247.8$$

**Problem-3:** **A harmonic oscillator of quality factor 10 is subjected to a sinusoidal applied force of frequency one and half times the natural frequency of the oscillator. If the damping be small, obtain (i) the amplitude of the forced oscillation in terms of its maximum amplitude and (ii) the angle by which it will be out of phase with the driving force.**

**Solution:** **The amplitude of forced vibration:**



$$A = \frac{F_0}{\sqrt{\left(\omega^2 - p^2\right)^2 + 4\lambda^2 p^2}}$$

$$\Rightarrow A = \frac{F_0}{\omega^2 \sqrt{\left(1 - \frac{p^2}{\omega^2}\right)^2 + \frac{4\lambda^2 p^2}{\omega^4}}}$$

**Now the quality factor:** $Q = \frac{\omega}{2\lambda} = 10 \Rightarrow \frac{2\lambda}{\omega} = \frac{1}{10}$

**and** $\frac{p}{\omega} = \frac{3}{2} \quad [\because p = 1\frac{1}{2}\omega]$

$$\therefore A = \frac{F_0}{\omega^2 \sqrt{\left(1 - \frac{p^2}{\omega^2}\right)^2 + \left(\frac{2\lambda}{\omega}\right)^2 \frac{p^2}{\omega^2}}}$$



$$= \frac{F_0}{\omega^2 \sqrt{\left(1 - \frac{3^2}{2^2}\right)^2 + \left(\frac{1}{10}\right)^2 \frac{3^2}{2^2}}} = 0.794 \frac{F_0}{\omega^2}$$

**And, for low damping, amplitude is maximum when** $p = \omega \Rightarrow \frac{p}{\omega} = 1$

$$\therefore A_{max} = \frac{F_0}{\omega^2 \sqrt{(1-1)^2 + \left(\frac{1}{10}\right)^2 1}} = 10\frac{F_0}{\omega^2}$$

**Therefore,** $$\frac{A}{A_{max}} = \frac{0.794}{10} = 0.08 \qquad \therefore A = 0.08 A_{max}$$

**(ii) Now the phase angle**

$$\tan\phi = \frac{2\lambda p}{\omega^2 - p^2} \qquad p = 1\frac{1}{2}\omega = \frac{3}{2}\omega$$



$$= \frac{\frac{\omega}{10}\cdot\frac{3\omega}{2}}{\omega^2 - \frac{9\omega^2}{4}^2} = -\frac{3}{25} \qquad Q = \frac{\omega}{2\lambda} \Rightarrow 2\lambda = \frac{\omega}{Q} = \frac{\omega}{10}$$

$$\therefore \phi = -6.84^0$$

**Therefore, the forced oscillation is (180° - 6.84°) = 173.2° out of phase with the driving force.**